

# Observing trajectories with weak measurements in quantum systems in the semiclassical regime

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## Abstract

We propose a scheme allowing to observe the evolution of a quantum system in the semiclassical regime along the paths generated by the propagator. The scheme relies on performing consecutive weak measurements of the position. We show how “weak trajectories” can be extracted from the pointers of a series of measurement devices having weakly interacted with the system. The properties of these “weak trajectories” are investigated and illustrated in the case of a time-dependent model system.

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In classical physics, the evolution of a physical system is given in terms of trajectories. Instead quantum mechanics forbids a fundamental description based on trajectories. Nevertheless the Feynman path integral approach gives a sum over paths formulation of the evolution of a quantum system, and when the actions are large relative to  $\hbar$  – the semiclassical regime –, the wavefunction evolves essentially along classical paths, those of the corresponding classical system [1]. Of course, this does not mean that a quantum object is a localized particle moving on a definite path. But trajectories may remain significant in quantum systems: the large scale properties, experimentally observed in many systems [1, 2], display the *signatures* of the underlying classical dynamics.

In this work, we aim to go further by proposing a scheme allowing to observe the evolution of a quantum system in the semiclassical regime along the trajectories of the corresponding classical system. The scheme relies on performing consecutive weak measurements (WM). WM [3, 4] are characterized by a very weak coupling between the system and the measurement apparatus. Thus measuring weakly an observable  $\hat{A}$  results in leaving the former essentially unperturbed while the latter picks up on average a limited amount of information encapsulated in the weak value

$$\langle \hat{A} \rangle_w = \frac{\langle \chi | \hat{A} | \psi \rangle}{\langle \chi | \psi \rangle}; \quad (1)$$

$|\psi\rangle$  is the initial (‘preselected’) state and  $\langle \chi |$  is the final (‘postselected’) state obtained by performing a standard strong measurement after having measured  $\hat{A}$  weakly. WM are receiving increased attention, either as a technique for signal amplification [5] or as a tool to investigate fundamental problems, from a theoretical standpoint but also experimentally [6]. In particular, in a beautiful recent experiment [7] ‘average trajectories’ for photons deduced indirectly from the WM of momentum have been observed. In our scheme we introduce instead ‘weak trajectories’ (WT) by measuring *directly* the position of the wavefunction interacting weakly with a set of measuring apparati. We will see that in the semiclassical regime the only WT compatible with the positions of the pointers are the classical paths.

Let  $|\psi(t_i)\rangle$  be the initial state of a dynamical system whose evolution is governed by a (possibly time-dependent) Hamiltonian  $H(t)$ . Let  $|\phi_k\rangle$  be the initial state of a measurement apparatus with  $\mathbf{R}_k$  indicating the position of the pointer. The meter wavefunction  $\langle \mathbf{R}_k | \phi_k \rangle$  is tightly localized around some position  $\mathbf{R}_k^0$ ; for convenience we choose a Gaussian wavefunction  $(2/\pi\Delta^2)^{1/2} e^{-(\mathbf{R}_k - \mathbf{R}_k^0)^2/\Delta^2}$  (we work from now on in a 2D configuration space and use

atomic units throughout). The local coupling between the meter and the system is assumed to take place during a small time interval  $\tau_k$ . The time-integrated interaction is taken as  $I_k = g\mathbf{r} \cdot \mathbf{R}_k \theta((4\Delta)^2 - |\mathbf{r} - \mathbf{R}_k|^2)$ , where  $g$  is the effective coupling strength and the last term is a unit-step function accounting for the short range character of the interaction (this term will be implicit in the rest of the paper). Assume now we have a set of measurement apparatus  $k = 1, \dots, n$  each positioned around a preassigned  $\mathbf{R}_k^0$ . The initial state of the system and measuring devices  $|\Psi(t_i)\rangle = |\psi(t_i)\rangle \prod_{k=1}^n |\phi_k\rangle$  evolves at time  $t_f$  to

$$|\Psi(t_f)\rangle = \exp -i [U(t_f, t_n) + I_n + \dots + I_2 + U(t_2, t_1) + I_1 + U(t_1, t_i)] |\psi(t_i)\rangle \prod_{k=1}^n |\phi_k\rangle \quad (2)$$

where  $U(t_{k+1}, t_k) = \int_{t_k}^{t_{k+1}} H(t') dt'$  is the self evolution of the system and  $t_k$  is taken as the interaction time (ie  $\tau_k \ll t_{k+1} - t_k$ ).

At time  $t_f$  a standard projective measurement is made in order to postselect the system to a desired final state  $|\chi(t_f)\rangle$ . Expanding each  $I_k$  in Eq. (2) to first order in the coupling  $g$  leads to

$$\prod_{k=1}^n \langle \mathbf{R}_k | \langle \chi(t_f) | \Psi(t_f) \rangle \simeq \langle \chi(t_f) | \psi(t_f) \rangle \prod_{k=1}^n \exp [-ig \langle \mathbf{r}(t_k) \rangle_W \cdot \mathbf{R}_k] \phi_k(\mathbf{R}_k, \mathbf{R}_k^0) \quad (3)$$

where  $\langle \mathbf{r}(t_k) \rangle_W$  is the weak value [Eq. (1)] given here by

$$\langle \mathbf{r}(t_k) \rangle_W \equiv \frac{\langle \chi(t_k) | \mathbf{r} | \psi(t_k) \rangle}{\langle \chi(t_k) | \psi(t_k) \rangle} \quad (4)$$

with  $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ . Eqs. (3)-(4) indicate that as a result of the interaction that took place at  $t_k$ , each meter wavefunction  $\phi_k(\mathbf{R}_k, \mathbf{R}_k^0)$  will incur a phase-shift given by the weak value  $\langle \mathbf{r}(t_k) \rangle_W$ . As in the standard WM scenario [3], this phase-shift appears as a shift in the momentum space of each meter. Since Eq. (3) holds provided  $g$  and  $\Delta$  are very small, the momentum space meter wavefunction will be broad, meaning a high number of events must be recorded in order to observe each shift.

The structure of  $\langle \mathbf{r}(t_k) \rangle_W$  deserves a special comment. Each  $\langle \mathbf{r}(t_k) \rangle_W$  is defined at  $t = t_k$  with an effective preselected state  $|\psi(t_k)\rangle = U(t_k, t_i) |\psi(t_i)\rangle$  being the initial state propagated *forward* in time and an effective postselected state  $\langle \chi(t_k) | = \langle \chi(t_f) | U(t_f, t_k)$  being

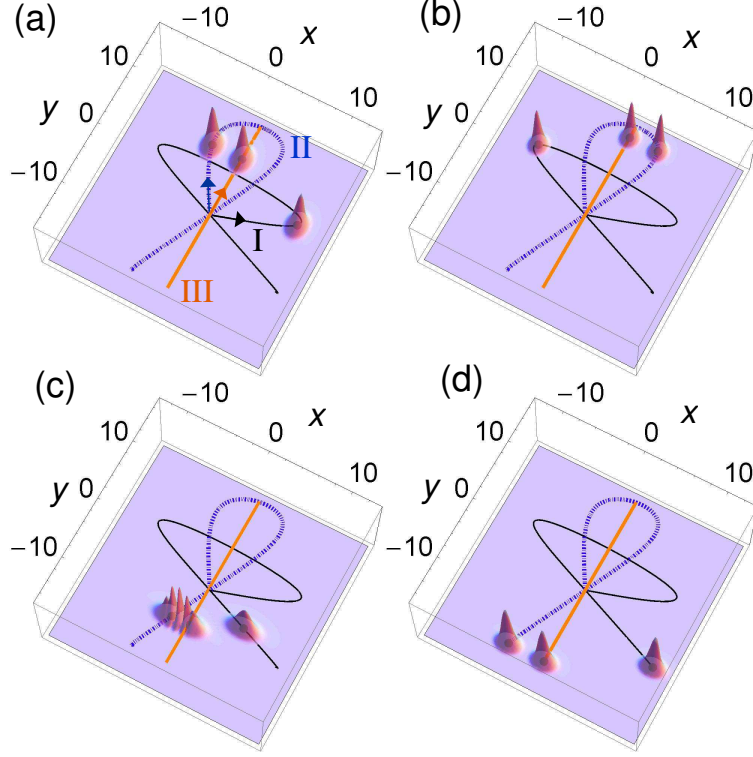


FIG. 1: Time evolution of the wavefunction initially ( $t_i = 0$ ) given by Eq. (8) with  $\mathbf{r}_0 = \mathbf{0}$  and the initial mean momenta  $\mathbf{p}_j, j = I, II, III$  taken as indicated by the arrows in panel (a). The reference classical trajectories  $I, II$  and  $III$  are shown resp. in black, dashed blue, and orange. (a) shows the wavefunction at  $t_1 = 0.7$ , (b) at  $t_2 = 2$ , (c) at  $t_3 = 3.15$  (after the wavepackets cross the origin) and (d) at  $t_f = 3.65$ , the time at which postselection is made [8].

the postselected state evolved *backward* in time; this property illustrates the close relation between WM and time-symmetric formulations of quantum mechanics [4]. Note that contrary to the usual definition of weak values, the effective pre and postselected states defining  $\langle \mathbf{r}(t_k) \rangle_W$  cannot be chosen: only the initial and the final states can be freely set. A weak value at some intermediate time  $t_k$  reflects the interaction  $I_k$  with the  $k$ th meter given the unitary evolution of the preselected and postselected states of the system. We can therefore envisage the set  $\{t_k, \langle \mathbf{r}(t_k) \rangle_W\}$  as defining a *weak trajectory* of the system evolving from an initial state to a final postselected state as recorded by the meter devices positioned at  $\mathbf{R}_k^0$ ,  $k = 1, \dots, n$ .

For an arbitrary quantum system a WT will typically reflect the space-time correlation between the forward evolution of the preselected state and the backward evolution of the

postselected state at the positions  $\mathbf{R}_k^0$  of the measurement apparati. Although obtaining this type of information is certainly of interest in general quantum systems, the notion of weak trajectories is particularly suited to investigate the evolution of a quantum system in the semiclassical regime. In this regime a typical wavefunction evolves according to the asymptotic form of the path integral propagator [9],

$$\psi(\mathbf{r}, t) = \int d\mathbf{r}' \left\{ \sum_{cl} \frac{1}{(2i\pi\hbar)} \left| \det \frac{\partial^2 S_{cl}(\mathbf{r}, \mathbf{r}', t)}{\partial \mathbf{r} \partial \mathbf{r}'} \right|^{1/2} \exp(iS_{cl}(\mathbf{r}, \mathbf{r}', t)/\hbar - i\mu_{cl}) \right\} \psi(\mathbf{r}', 0) \quad (5)$$

where  $cl$  runs on the classical trajectories connecting  $\mathbf{r}'$  to  $\mathbf{r}$  in time  $t$  (from now on we set  $t_i = 0$ ) and the term between  $\{\dots\}$  is the semiclassical propagator obtained from the asymptotics ( $S_{cl} \gg \hbar$ ) of the path integral form of the evolution operator  $U(t, 0)$ .  $S_{cl}$  is the classical action and  $\mu_{cl}$  the topological index of each path. Working out the full semiclassical propagation is often a formidable task, especially as the number of trajectories proliferate in the regimes where the semiclassical approximation holds. However if the initial state is well localized, the semiclassical propagation can be simplified by linearizing the action [10] around an initial and a final reference point linked in time  $t$  by a central classical trajectory, the guiding trajectory. Linearization is particularly relevant if  $\psi(\mathbf{r}', 0)$  is chosen to be a localized Gaussian

$$\psi_{\mathbf{r}_0, \mathbf{p}_0}(\mathbf{r}, 0) = \left( \frac{2}{\pi\delta^2} \right)^{1/2} e^{-(\mathbf{r}-\mathbf{r}_0)^2/\delta^2} e^{i\mathbf{p}_0 \cdot (\mathbf{r}-\mathbf{r}_0)/\hbar}. \quad (6)$$

The initial reference point is the maximum of the Gaussian, the linearized action in Eq. (5) is a quadratic form, whereas the determinant prefactor becomes a purely time-dependent term that can be written in terms of the stability matrix of the guiding trajectory. This linearized information along the central reference trajectory effectively replaces the sum over  $cl$ .

The integral (5) can then be performed exactly: the result is of the form [10]

$$\psi_{\mathbf{r}_0, \mathbf{p}_0}(\mathbf{r}, t) = \text{Tr} [\mathbf{A}(t)] e^{-(\mathbf{r}-\mathbf{q}(t)) \cdot [\mathbf{M}(t) + i\mathbf{N}(t)] \cdot (\mathbf{r}-\mathbf{q}(t))} e^{i\mathbf{p}(t) \cdot (\mathbf{r}-\mathbf{q}(t))/\hbar} e^{iS(\mathbf{q}(t), \mathbf{r}_0, t)/\hbar} \quad (7)$$

where  $(\mathbf{q}(t), \mathbf{p}(t))$  are the phase-space coordinates of the guiding classical trajectory with initial conditions  $(\mathbf{r}_0, \mathbf{p}_0)$  and  $\mathbf{A}$ ,  $\mathbf{M}$  and  $\mathbf{N}$  are time-dependent matrices depending solely on

the stability elements of the guiding trajectory. The advantage of working in the linearized regime is that by picking a preselected state of the form

$$\langle \mathbf{r} | \psi(t_i) \rangle = \sum_j c_j \psi_{\mathbf{r}_0, \mathbf{p}_j}(\mathbf{r}, t_i) \quad (8)$$

i.e. a superposition of Gaussians (6) launched in different directions  $\mathbf{p}_j$ , one is dealing conceptually with the type of problem defined by the semiclassical propagation (5) with a simplified and perfectly controlled dynamics. The evolution operator  $U(t_1, t_i)$  of Eq. (2) propagates each term of Eq. (8) along the relevant guiding trajectory yielding at  $t_1$  the superposition of evolved states each given by Eq. (7)  $\sum_j c_j \psi_{\mathbf{r}_0, \mathbf{p}_j}(\mathbf{r}, t_1)$  (recall that  $\mathbf{q}, \mathbf{p}$  and the matrices  $\mathbf{A}, \mathbf{M}$  and  $\mathbf{N}$  explicitly depend on  $j$ ).

The postselected state will also be taken to be a Gaussian of the type (6) localized in the vicinity of a chosen point  $\mathbf{r}_f$  at time  $t_f$

$$\chi_{\mathbf{r}_f, \mathbf{p}_f}(\mathbf{r}, t_f) = \left( \frac{2}{\pi \delta_f^2} \right)^{1/2} e^{-(\mathbf{r}-\mathbf{r}_f)^2 / \delta_f^2} e^{i \mathbf{p}_f \cdot (\mathbf{r}-\mathbf{r}_f) / \hbar}. \quad (9)$$

In the linearized approximation, finding the backward propagated state is tantamount to obtaining the *unique* solution of the form (7) such that  $\psi_{\mathbf{r}(t_1), \mathbf{p}(t_1)}(\mathbf{r}, t_f) = \chi_{\mathbf{r}_f, \mathbf{p}_f}(\mathbf{r}, t_f)$ : this gives a wavefunction centered on the time-reversed classical trajectory having boundary conditions  $(\mathbf{r}_f, \mathbf{p}_f)$  at  $t = t_f$ . As illustrated below, suppose that the measurement apparati are all localized along the guiding trajectory followed by one of the branches  $j$  of the preselected state, so that the positions obey  $\mathbf{R}_k^0 = \mathbf{q}_j(t_k)$ . Assume for the moment that all these  $\mathbf{R}_k^0$  lie at positions where the overlap between the different branches of the system wavefunction is negligible. The weak values (4) can then be computed exactly. Two situations are then possible. (i) If  $\chi_{\mathbf{r}_f, \mathbf{p}_f}(\mathbf{r}, t_k) \approx 0$  in the vicinity of  $\mathbf{R}_k^0$  the meter does not move: there is no weak trajectory in the neighborhood of this point. (ii) Otherwise, writing the distance between the meter and the position at time  $t_k$  of the backward evolved trajectory as  $\boldsymbol{\epsilon}_k = \mathbf{R}_k^0 - \mathbf{q}_f(t_k)$ , the weak value  $\langle \mathbf{r}(t_k) \rangle_W = \langle x(t_k) \rangle_W \hat{\mathbf{x}} + \langle y(t_k) \rangle_W \hat{\mathbf{y}}$  takes the form

$$\begin{aligned} \langle x(t_k) \rangle_W &= [\mathbf{R}_k^0 \cdot \hat{\mathbf{x}} + a_x \boldsymbol{\epsilon}_k \cdot \hat{\mathbf{x}} + b_x (\mathbf{p}_j(t_k) - \mathbf{p}_f(t_k)) \cdot \hat{\mathbf{x}}] \\ &+ i [g_x(t_k) \boldsymbol{\epsilon}_k \cdot \hat{\mathbf{x}} + h_x (\mathbf{p}_j(t_k) - \mathbf{p}_f(t_k)) \cdot \hat{\mathbf{x}}] \end{aligned} \quad (10)$$

and analog expressions for  $\langle y(t_k) \rangle_W$ .  $a, b, g$  and  $h$  are time-dependent functions whose explicit forms are cumbersome though straightforward to evaluate.

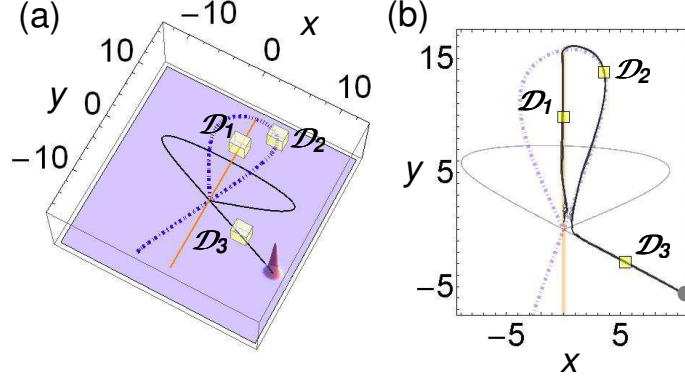


FIG. 2: (a) The postselected wavefunction, a Gaussian localized on the guiding trajectory  $I$ , is shown, along with the positions of the measuring devices  $\mathcal{D}_{1,2,3}$ . Only the pointer in  $\mathcal{D}_3$  is affected, while  $\mathcal{D}_{2,3}$  remain still: there is no WT joining the corresponding positions. (b) The ‘average trajectories’ obtained from WM of the *momentum* (repeated at successive positions) are plotted in solid black: 9 trajectories having their final positions on and near the maximum of the postselected state are shown, along with the reference trajectories (in faded colors).

The structure of Eq. (10) emphasizes the special rôle of the postselected state with  $(\mathbf{r}_f, \mathbf{p}_f)$  chosen such that the backward evolved trajectory simply retraces the guiding trajectory  $\mathbf{q}_j(t)$ . For this special choice  $\epsilon_k = 0$  and  $\mathbf{p}_j(t_k) = \mathbf{p}_f(t_k)$  for each  $k$ , so that *each* meter records its own position,  $\langle \mathbf{r}(t_k) \rangle_W = \mathbf{R}_k^0$ . The WT  $\{t_k, \langle \mathbf{r}(t_k) \rangle_W\}$  recorded by the meters thus corresponds to the guiding trajectory of the linearized Feynman propagator. For any other choice of postselection,  $\langle \mathbf{r}(t_k) \rangle_W$  can either be undefined (for those  $k$  such that  $\chi_{\mathbf{r}_f, \mathbf{p}_f}(\mathbf{r}, t_k) \approx 0$ ) or yield a complex number depending on  $\epsilon_k$  and the difference in momenta between the forward propagated preselected and backward propagated postselected states. The WT (where defined) becomes a list of complex numbers correlating the forward and backward evolution at points  $\mathbf{R}_k^0$ . Essentially the same situation is found when the  $\mathbf{R}_k^0$  are not placed along a guiding classical trajectory.

For the purpose of illustration – and to avoid spurious effects due to the quality of the linearized approximation –, we will take a 2D time-dependent linear oscillator (TDLO). The linearized propagator for the TDLO is quantum mechanically exact, while the varying amplitudes capture many features of semiclassical systems with more involved dynamics. The Hamiltonian for the system is  $H = (P_x^2 + P_y^2)/2m + mV_x(t)x^2 + mV_y(t)y^2$  where for definiteness we choose  $V_i(t) = \kappa_i - v_i \cos(2\omega_i t)$  ( $i = x, y$ ;  $\kappa, v$  and  $\omega$  are constants). The

wavefunction (7) is obtained directly by employing standard path integral techniques [9]; the classical trajectories can be found in closed form from the solutions of Ermakov systems [11, 12]. The preselected state (8) is taken as the superposition of 3 Gaussians at the origin with mean momenta as shown in Fig. 1(a). The maximum of each wavepacket then evolves by following the guiding trajectory, *I*, *II* or *III* as shown in Fig. 1.

Let us first set the postselected state (9) with  $\mathbf{r}_f = \mathbf{q}_I(t_f)$  and  $\mathbf{p}_f = \mathbf{p}_I(t_f)$  and let us position the measuring devices  $\mathcal{D}_k$  as shown in Fig. 2(a). The backward evolution of  $|\chi(t_f)\rangle$  simply retraces trajectory *I* backwards. Therefore the pointer in  $\mathcal{D}_3$  yields according to Eq. (10) the position  $\mathbf{R}_3^0$  while  $\mathcal{D}_2$  and  $\mathcal{D}_1$  do not move at all (no overlap with  $|\chi(t)\rangle$  at any  $t$ ). One concludes that the ‘particle’ went through  $\mathcal{D}_3$  but not through  $\mathcal{D}_1$  and  $\mathcal{D}_2$ . If instead of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  other devices  $\mathcal{D}'_1$  and  $\mathcal{D}'_2$  positioned as shown in Fig. 3(a) are employed, then all the meters yield their respective positions as the weak outcomes: the ‘particle’ went through  $\mathcal{D}'_1, \mathcal{D}'_2$  and  $\mathcal{D}_3$ . Hence one concludes (possibly by inserting additional devices) that the ‘particle’ took the WT defined by the classical trajectory *I*. Note that according to Eq. (10) there is no quantum state of the form (9) that can yield a WT going through  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  and  $\mathcal{D}_3$ . This is due to the fact that there does not exist a wavepacket arriving in the neighborhood of  $\mathbf{r}_f$  at time  $t_f$  that would have previously visited the neighborhoods of  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  and  $\mathcal{D}_3$ .

The last remark highlights the incompatibility between the ‘weak trajectories’ defined here by consecutive WM of the position and the ‘average trajectories’ (AT) defined by a WM of the *momentum* immediately postselected to a given position. By repeating these weak momentum measurements for different postselected positions, a velocity field is obtained. The AT are precisely the trajectories built on this velocity field. They have been experimentally observed recently for photons in a double slit setup [7]. It was previously known [13] that their dynamics is governed by the law of motion of the de Broglie-Bohm theory, i.e. by the probability flow, whereas the WT are generated by the semiclassical propagator (5). The mismatch [14] between de Broglie-Bohm and classical trajectories in semiclassical systems hinges on the fact that when wavepackets interfere, the overall mean velocity field differs from the group velocity of each individual wavepacket. The mismatch is illustrated here in Fig. 2(b): we have computed numerically [12] the AT arriving in the neighborhood of  $\mathbf{r}_f = \mathbf{q}_I(t_f)$ . These AT go indeed through  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  and  $\mathcal{D}_3$ : starting near the origin, they first move in the vicinity of the guiding trajectory *III*, then travel along



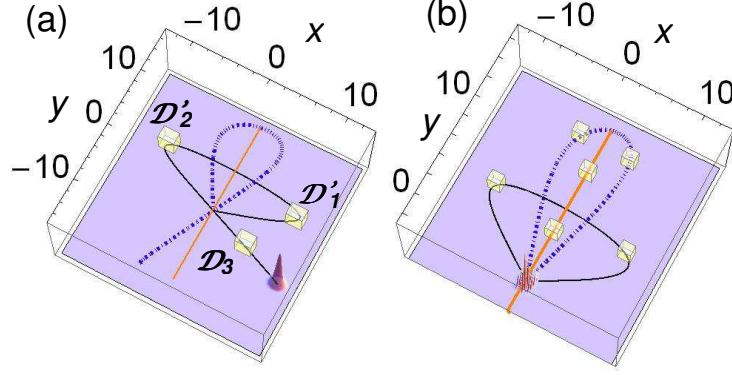


FIG. 3: (a): The postselected wavefunction (the same as displayed in Fig. 2) is shown with the measuring devices  $\mathcal{D}'_1, \mathcal{D}'_2$  and  $\mathcal{D}_3$ . All these pointers indicate their own position: the evolution of the system along the reference trajectory  $I$  has been measured weakly. (b): Postselection now takes places at  $t_O = 2.84$  when the wavepackets return simultaneously to the origin. The postselected state, chosen as a superposition of Gaussians localized around the origin with mean momenta pointing along the guiding trajectories is plotted along with measuring devices positioned along the reference trajectories  $I, II, III$ . All the meters have moved: the semiclassical sum over paths formulation has thus been measured weakly.

trajectory  $II$  and thereafter ‘jump’ so as to move along trajectory  $I$ .

Finally, consider choosing postselection at  $t_f = t_O$  when trajectories  $I, II$  and  $III$  first return to the origin, with the postselected state chosen as the superposition  $\chi_O(\mathbf{r}, t_O) = \sum_j \chi_{\mathbf{r}_f=0, \mathbf{p}_{Oj}}(\mathbf{r}, t_O)$ , with  $\chi_{\mathbf{r}_f=0, \mathbf{p}_{Oj}}$  given by Eq. (9) and  $\mathbf{p}_{Oj} = \mathbf{p}_j(t_O)$ ,  $j = I, II, III$ . Several measuring devices are positioned as shown in Fig. 3(b). By construction the backward evolution of  $\chi_O$  yields a superposition of wavepackets retracing trajectories  $I, II$  and  $III$  respectively. Therefore *all* the pointers will display their own position as the weak value indicating the ‘particle’ was there. This is an experimentally realizable way to catch the essence of the path integral approach in the semiclassical regime: weakly interacting measuring devices indicate the ‘particle’ takes *simultaneously* all the available classical paths. In contrast a strong projective measurement would of course yield a definite outcome on only *one* of the paths.

To sum up, we have defined ‘weak trajectories’ allowing to observe the paths taken by a quantum system in the semiclassical regime by direct weak measurements of the position. A consequence worth exploring concerns the possibility of employing this scheme to reconstruct

the unknown propagator of a semiclassical system from the observed weak trajectories. The present setup may also be used in designing pre-post selected quantum paradoxes containing dynamical ingredients.

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